Computing Travelling Flexural-Gravity Waves

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Acknowledgements

This is joint work with

- **Paul Milewski** at University of Bath
- **Emilian Părău** at University of East Anglia
- **Jean-Marc Vanden-Broeck** at University College London

Outline

Outline

Goal

Efficiently compute solutions for different models for waves under ice (flexural-gravity waves) and compare the solutions.

Waves Under Ice Generated by a Moving Truck[1]

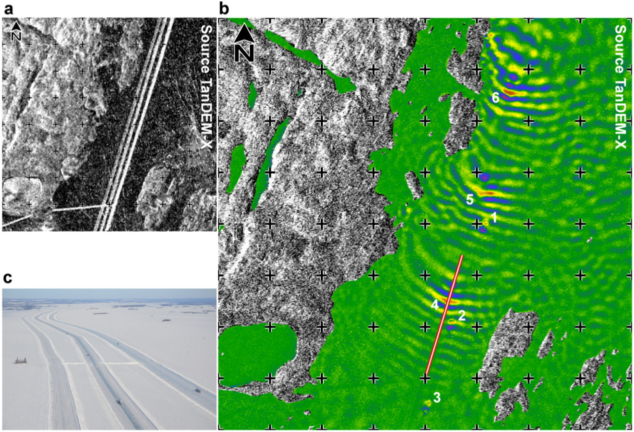


Figure: Waves generated by transport trucks.

[1] J.J. van der Sanden and N.H. Short, *"Radar satellites measure ice cover displacements induced by moving vehicles"*, Cold Regions Science and Technology, 133, 56-62 (2017)

Tsunami Under Ice[2]

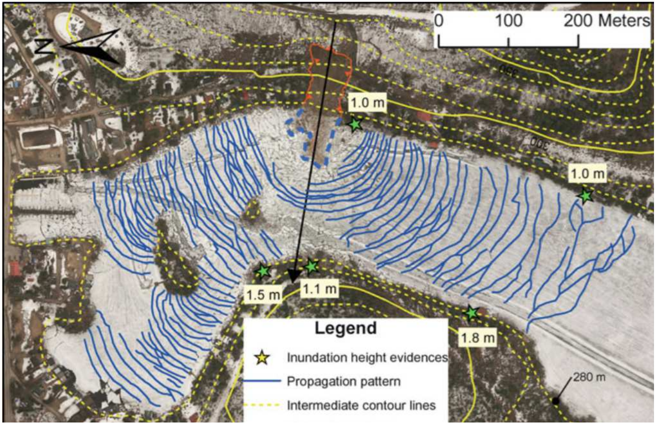


Figure: Observations of coastal landslide-generated tsunami under an ice cover in Quebec

[2] J. Leblanc et al, *"Observations of Coastal Landslide-Generated Tsunami Under an Ice Cover: The Case of Lac-des-Seize-Îles, Québec, Canada"* Submarine Mass Movements and their Consequences, pp. 607-614, (2016)

Outline

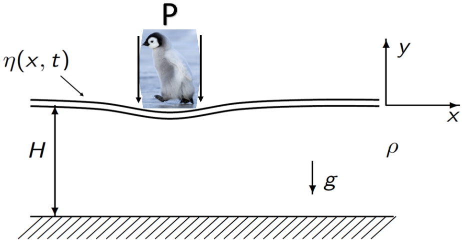
Model for Water Waves

For an inviscid, incompressible fluid with velocity potential $\phi(x, y, z, t)$, the forced Euler's equations are given by

$$\begin{cases} \triangle\phi = 0, & (x, y, z) \in \Omega, \\ \phi_z = 0, & z = -h, \\ \eta_t + \eta_x\phi_x + \eta_y\phi_y = \phi_z, & z = \eta(x, y, t), \\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{F^2}\eta + P(x, y, t) = -D\frac{\delta H}{\delta\eta}, & z = \eta(x, y, t), \end{cases}$$

where
h: depth
$F = \frac{c}{\sqrt{gh}}$: Froude number
D: flexural rigidity
$\eta(x, y, t)$: variable surface
$P(x, y, t)$: external pressure distribution
$\frac{\delta H}{\delta\eta}$: condition at the interface.
Ω: either periodic or infinite in x and y

Conditions at the Interface



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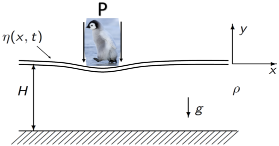

The term modelling the ice assumes

- Thin elastic plate with constant thickness
- The ice bends with the water waves
- No friction between the ice and the water
- Continuous sheet, no breaking
- No shear

with the coefficient for flexural rigidity D given by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

with E: Young's modulus, ν: Poisson ratio, h: thickness of the ice.

Models For a Thin Sheet of Ice

We consider two models

- Biharmonic (linear) model, assuming ice behaves like an Euler-Bernoulli thin elastic plate that gets deflected by a load (regime where curvature is small)

$$H_L = \frac{1}{2}\int (\triangle\eta)^2 dA$$

- Cosserat (nonlinear) model, assuming the sheet of ice can bend, twist and stretch (has a Willmore energy) [3]

$$H_N = \frac{1}{2}\int (\kappa_1 + \kappa_2)^2 dS \quad \text{with } \kappa_1, \kappa_2 \text{ principle curvatures}$$

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[3] Plotnikov and Toland, *"Modelling nonlinear hydroeslastic waves"*, Phil. Trans. R. Soc. 369, 1942-2956 (2011)

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Background

A lot of work on this topic, here are a select few

- **Modelling ice**: Since Greenhill (1886), people have been deriving linear and nonlinear elasticity models with some that conserve energy (for example Plotnikov-Toland/Cosserat model) and some that don't (for example Kirchoff-Love model) For a review see Squire et at. (2007)
- **Existence of Solutions**:
- **Solutions in two dimensions**:
- **Solutions in three dimensions**:
- **High performance computing techniques**:

Background

A lot of work on this topic, here are a select few

- **Modelling ice**:
- **Existence of Solutions**: Existence for some parameters using Lagrangian formulation for travelling waves (Toland et al., 2008). Akers, Ambrose and Sulon prove existence and show bifurcation branches of solutions in 2D (2017).
- **Solutions in two dimensions**:
- **Solutions in three dimensions**:
- **High performance computing techniques**:

Background

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- **Modelling ice**:
- **Existence of Solutions**:
- **Solutions in two dimensions**:
  - Vanden-Broeck and Părău (2011) computed generalised solitary waves and periodic waves under an ice sheet using the Kirchhoff-Love model.
  - Gao and Vanden-Broeck (2014) numerically computed periodic and generalised solitary waves using Plotnikov-Toland model.
  - Solutions for gravity waves and capillary-gravity waves have been computed using AFM method by Deconinck, Oliveras and T.
- **Solutions in three dimensions**:
- **High performance computing techniques**:

Background

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- **Modelling ice**:
- **Existence of Solutions**:
- **Solutions in two dimensions**:
- **Solutions in three dimensions**:
  - Asymptotic models by Wang and Milewski (2013) show flexural-gravity solitary waves do not bifurcate from zero amplitude solution.
  - Vanden-Broeck and Părău have been using the BIM method to compute three dimensional waves for gravity, capillary-gravity and the linear model for flexural-gravity waves
- **High performance computing techniques**:

Background

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- **Modelling ice**:
- **Existence of Solutions**:
- **Solutions in two dimensions**:
- **Solutions in three dimensions**:
- **High performance computing techniques**: Pethiyagoda et al. (2014) computed small amplitude solutions for wake patterns using Krylov methods, using a preconditioner based on the linearisation.

Outline

Methods

There is a variety of methods for reformulating the problem. We focus on

1. Boundary Integral Method (BIM) (1989) based on work by Forbes.
2. Ablowitz, Fokas and Musslimani method (AFM) (2006)

Both of these methods have their advantages and disadvantages. The main two disadvantages are

- Small denominators in the integrands for BIM
- Exponentially large terms in the integrands for AFM

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Identity behind BIM

Use Green's second identity

$$\int_V (\alpha \Delta \beta - \beta \Delta \alpha) dV = \oint_{S(V)} \left(\alpha \frac{\partial \beta}{\partial n} - \beta \frac{\partial \alpha}{\partial n} \right) dS$$

where in three dimensions, β is the fundamental solution given by

$$\frac{1}{4\pi} \frac{1}{((x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2)^{1/2}}$$

and $\alpha = \phi - x$, which satisfies Laplace's equation.

Identity behind AFM

If both ϕ and ψ satisfy Laplace's equation, then

$$(\phi_z\psi_x + \psi_z\phi_x)_x + (\phi_z\psi_y + \psi_z\phi_y)_y + (\phi_z\psi_z - \phi_x\psi_x - \phi_y\psi_y)_z = 0$$

Let $\psi(x, y, z) = e^{ik_1x+ik_2y+ikz}$ be a particular solution with $k = \sqrt{k_1^2 + k_2^2}$.
Use the divergence theorem and the boundary as well as conditions on the solutions to obtain the non-local equation.

Surface Variables

Reformulate into surface variables (Zakharov 1969)

$$q(x, y, t) = \phi(x, y, z = \eta, t)$$

Using chain rule,

$$\phi_x = \frac{(1 + \eta_y^2)q_x - \eta_x \eta_y q_y - \eta_x \eta_t}{1 + |\nabla \eta|^2}$$

$$\phi_y = \frac{(1 + \eta_x^2)q_y - \eta_x \eta_y q_y - \eta_y \eta_t}{1 + |\nabla \eta|^2}$$

$$\phi_z = \frac{\eta_x q_x + \eta_y q_y + \eta_t}{1 + |\nabla \eta|^2}$$

Then the Bernoulli condition (local equation) becomes

$$q_t + \frac{1}{2}\left|\nabla q\right|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)} = -D\frac{\delta H}{\delta \eta}$$

Outline

Reformulation

Starting with Euler's equations

- In two dimensions, the local equation is given by

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\frac{(\eta_t + \eta_x q_x)^2}{1+\eta_x^2} = -D\frac{\delta H}{\delta \eta}.$$

- In two dimensions, the nonlocal equation [4] is given by

$$\int_0^{2\pi} e^{ikx}\left(i\eta_t \cosh(k(\eta+h)) + q_x \sinh(k(\eta+h))\right) dx = 0,$$

$$\forall k \in \mathbb{Z},\ \ k \neq 0.$$

[4] Ablowitz, Fokas and Musslimani, *"On a new non-local formulation of water waves"*, J. Fluid Mech., vol. 562, pp. 313343 (2006)

Reformulation

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Reformulation

- Switching to the travelling frame by setting $(x, t) \to (x - ct, t)$.
- Looking at the steady-state problem, set $\eta_t = q_t = 0$.
- Use the local equation to obtain q_x.
- The non-local equation becomes

$$\int_0^{2\pi} e^{ikx} \sqrt{(1+\eta_x^2)\left(c^2 - 2g\eta - 2D\frac{\delta H}{\delta \eta}\right)} \sinh(k(\eta + h))dx = 0.$$

$$\forall k \in \mathbb{Z}, \;\; k \neq 0.$$

where $\frac{\delta H}{\delta \eta}$ for the linear model is

$$\frac{\delta H}{\delta \eta} = \eta_{4x}$$

and for the nonlinear model

$$\frac{\delta H}{\delta \eta} = \frac{1}{(1+\eta_x^2)}\partial_x\left[\frac{1}{(1+\eta_x^2)}\partial_x\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)\right] + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3$$

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Numerical Continuation

Recall

$$\int_0^{2\pi} e^{ikx} \sqrt{(1+\eta_x^2)\left(c^2 - 2g\eta - 2D\frac{\delta H}{\delta \eta}\right)} \sinh(k(\eta + h))dx = 0.$$

We want to generate a bifurcation diagram:

1. Assume in general $\eta_N(x) = \sum_{j=1}^{N} a_j \cos(jx)$.
2. Linearizing we can find the bifurcation will start when $c = \sqrt{(g+\sigma)\tanh(h)}$ and $\eta(x) = a\cos(x)$.
3. Use this guess in Newton's method to compute the true solution.
4. Scale the previous solution to get a guess for the new bifurcation parameter.
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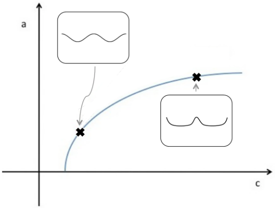
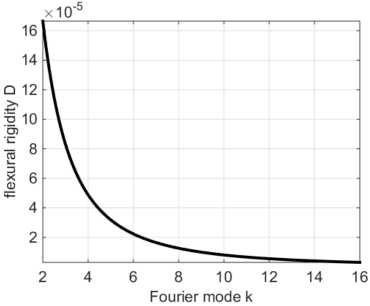


Resonance

At the bifurcation point, the resonance condition is given by

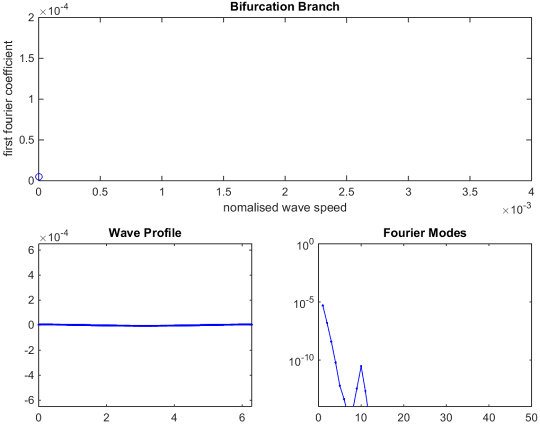
$$(g + D)K\tanh(h) - \left(g + K^4 D\right)\tanh(Kh) = 0. \qquad (K \neq 1).$$



then we obtain the equivalent of Wilton ripples.

Flexural-Gravity waves: Resonant Solutions at $k = 10$

$$h = 0.05 \text{ and } D \approx 8.1085 \times 10^{-5}$$



Comparing Models in Infinite Depth

Bifurcation branches change direction depending on flexural rigidity D and can differ for different models

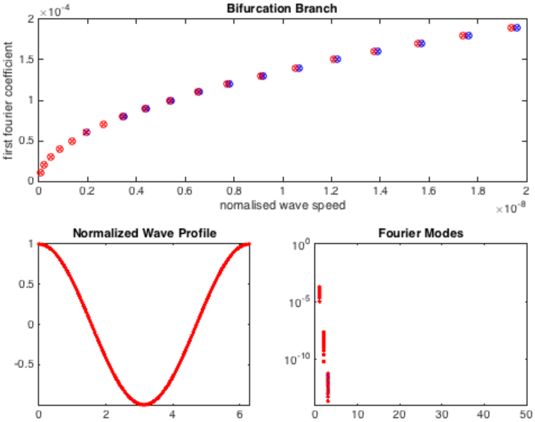


Figure: Small amplitude waves for the nonlinear model and linear model for ice with $D = 0.01$

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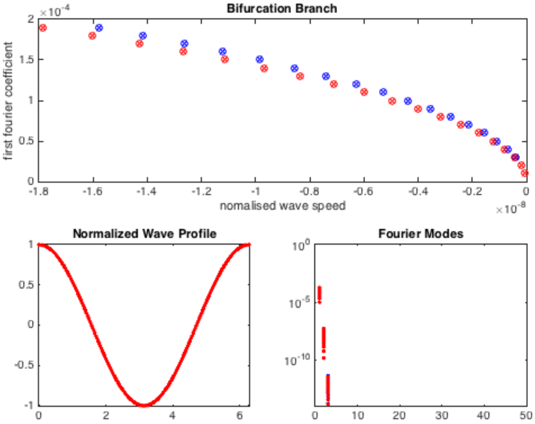

Figure: Small amplitude waves for the nonlinear model and linear model for ice with $D = 0.1$

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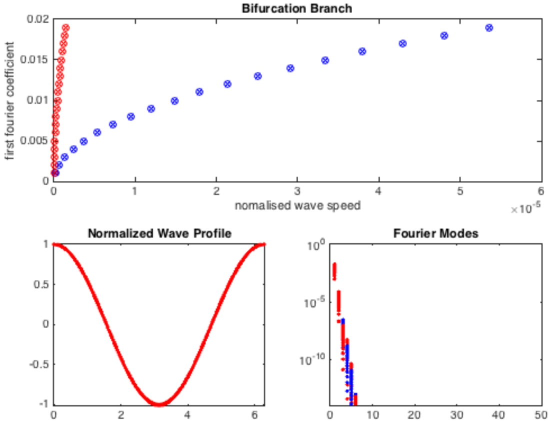

Figure: Small amplitude waves for the nonlinear model and linear model for ice with $D = 0.3$

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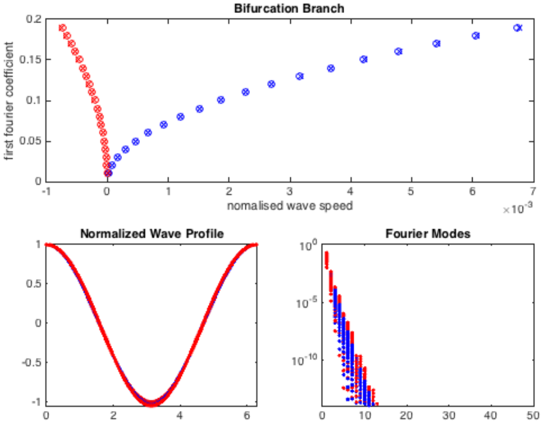
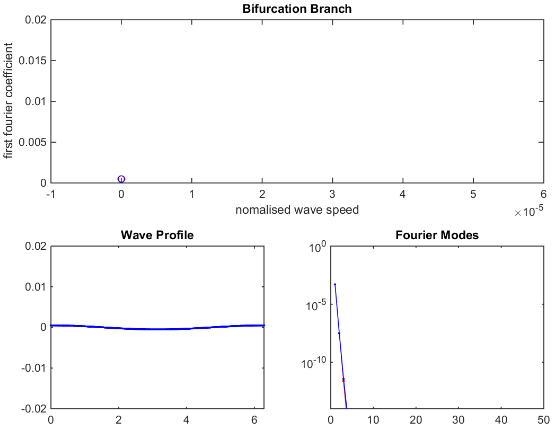


Figure: Small amplitude waves for the nonlinear model and linear model for ice with $D = 0.5$

Flexural-Gravity waves: Infinite Depth

Infinite depth with $D = 0.5$



Outline

Models for Ice

The two different models are considered

- Biharmonic (linear) model

$$\frac{\delta H}{\delta \eta} = \nabla^4 \eta$$

- Cosserat (nonlinear) model

$$\frac{\delta H}{\delta \eta} = \frac{2}{\sqrt{a}}\left[\partial_x\left(\frac{1+\eta_y^2}{\sqrt{a}}\partial_x H\right) - \partial_x\left(\frac{\eta_x\eta_y}{\sqrt{a}}\partial_y H\right) - \partial_y\left(\frac{\eta_x\eta_y}{\sqrt{a}}\partial_x H\right) + \partial_y\left(\frac{1+\eta_x^2}{\sqrt{a}}\partial_y H\right)\right]$$
$$+ 4H^3 - 4KH$$

where

$$a = 1 + \eta_x^2 + \eta_y^2$$
$$H = \frac{1}{2}a^{3/2}\left[(1+\eta_y^2)\eta_{xx} - 2\eta_{xy}\eta_x\eta_y + (1+\eta_x^2)\eta_{yy}\right]$$
$$K = \frac{1}{a^2}\left[\eta_{xx}\eta_{yy} - \eta_{xy}^2\right]$$

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$$K = \frac{1}{a^2} \left[\eta_{xx}\eta_{yy} - \eta_{xy}^2 \right]$$

System of Equations

The final form of equations to solve for flexural-gravity waves in infinite depth is

$$\frac{1}{2}\frac{(1+\eta_x^2)q_y^2+(1+\eta_y^2)q_x^2-2\eta_x\eta_y q_x q_y}{1+\eta_x^2+\eta_y^2}+\frac{\eta}{F^2}+P+D\frac{\delta H}{\delta\eta}=\frac{1}{2}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(q-q^*-x+x^*)K_1+\eta_x K_2\right]dxdy=2\pi(q^*-x^*)$$

where

$$K_1=\frac{1}{d^{3/2}}(\eta-\eta^*-(x-x^*)^2\eta_x-(y-y^*)^2\eta_y)$$
$$K_2=\frac{1}{d^{1/2}}$$

with

$$d(x,y,x^*,y^*,\eta)=(x-x^*)^2+(y-y^*)^2+(\eta-\eta^*)^2$$

.

Symmetry

Symmetry in y direction

$$\eta(x,y) = \eta(x,-y)$$

and

$$q(x,y) = q(x,-y)$$

implies additional terms

$$\frac{1}{2}\frac{(1+\eta_x^2)q_y^2+(1+\eta_y^2)q_x^2 - 2\eta_x\eta_y q_x q_y}{1+\eta_x^2+\eta_y^2} + \frac{\eta}{F} - \frac{1}{2} = F(\eta)$$

$$\int_0^\infty \int_{-\infty}^\infty \left[(q-q^* - x + x^*)\tilde{K}_1 + \eta_x \tilde{K}_2\right] dxdy = 2\pi(q^* - x^*)$$

where

$$\tilde{K}_1 = \bar{K}_1(x,y,\eta,x^*,y^*,\eta^*) + \bar{K}_1(x,-y,\eta,x^*,y^*,\eta^*)$$
$$\tilde{K}_2 = \bar{K}_2(x,y,\eta,x^*,y^*,\eta^*) + \bar{K}_2(x,-y,\eta,x^*,y^*,\eta^*)$$

Removing the Singularity

Part of the integral is singular [5]. Remove it by noting that

$$\int\int \eta_x \tilde{K}_2 dxdy = \int\int \left[\tilde{K}_2\eta_x - \eta_x^*\tilde{S}_2\right] dxdy + \eta_x^* \int\int \tilde{S}_2 dxdy$$

where

$$S_2 = \frac{1}{\sqrt{(1+\eta_x^{*2})(x-x^*)^2 + 2\eta_x^*\eta_y^*(x-x^*)(y-y^*) + (1+\eta_y^{*2})(y-y^*)^2}}$$

The integral in the box can be computed since it looks like $\int \frac{1}{z} dz = \ln z$.

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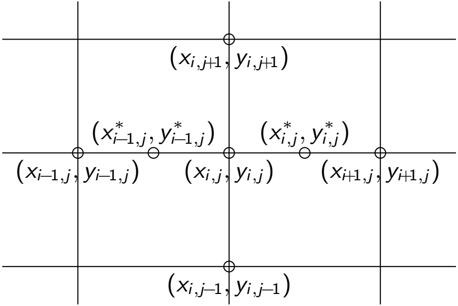
Discretisation

- Let x_i and y_j be equally spaced points such that $i = 1, \ldots, N$ and $j = 1, \ldots, M$.



- Let the vector of unknowns be $q_{x(i,j)}$ and $\eta_{x(i,j)}$ such that

$$u = \left[q_{x(1,1)}, \cdots, q_{x(N,1)}, \cdots, q_{x(N,M)}, \eta_{x(1,1)}, \cdots, \eta_{x(N,M)}\right]^T$$

- Use finite differences to discretise the derivatives
- Obtain 2NM equations

$$G(u) = 0$$

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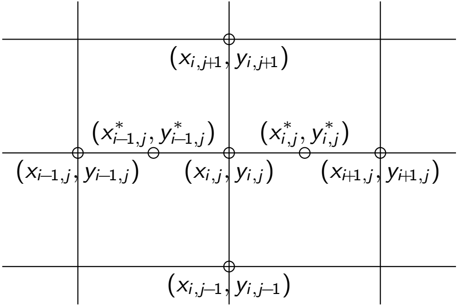

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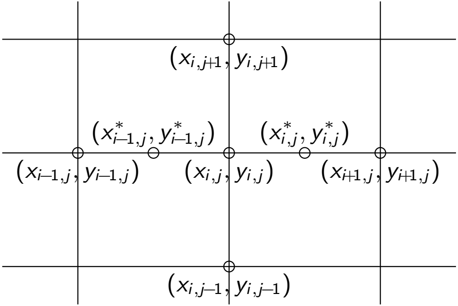

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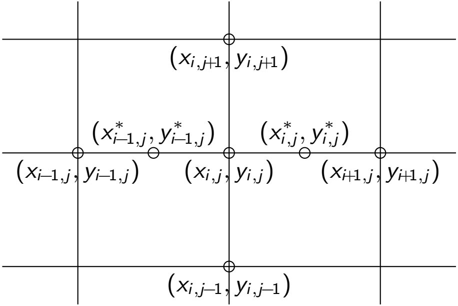

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Numerical Approach

To solve the system

1. Set up an initial guess u^0
2. Until convergence
   2.1 Solve $J(u^n)\delta^n = -G(u^n)$
   2.2 Set $u^{n+1} = u^n + \lambda\delta^n,\ 0 < \lambda < 1$
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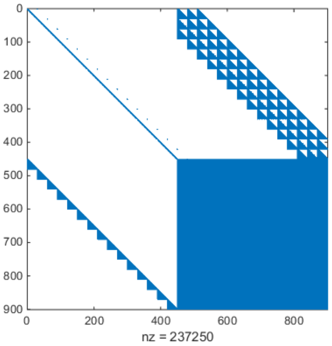
Jacobian

The sparsity of the linearised Jacobian for flexural-gravity waves



Solving the System of Equations

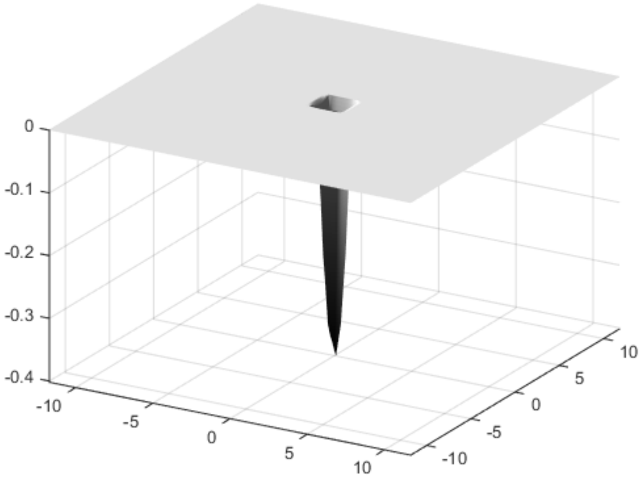
The most **computationally intensive part is computing the Jacobian**. We consider two ways of solving the system of equations

1. Inexact Newton Method: (direct method) uses an inexact Jacobian (not computed at each step).
2. Modified Newton Method: (iterative method) using a preconditioned Krylov method to construct the solution.
   - Can use the Jacobian for some previous iterate as a preconditioner.

Note: completely matrix-free methods can't be used since the Jacobian is not a sparse matrix

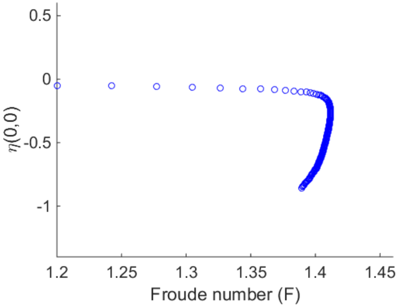
Forcing Term

We use the following pressure as a forcing for depression waves



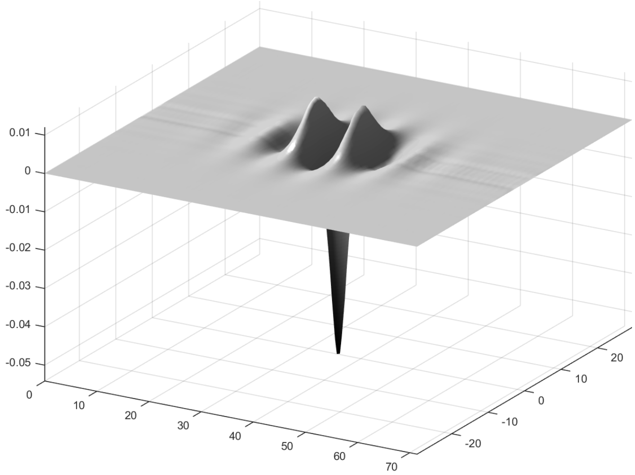
Sample Bifurcation Branch

Forced depression waves using the nonlinear model for ice



Sample Solutions

Solutions for forced waves underneath an ice sheet



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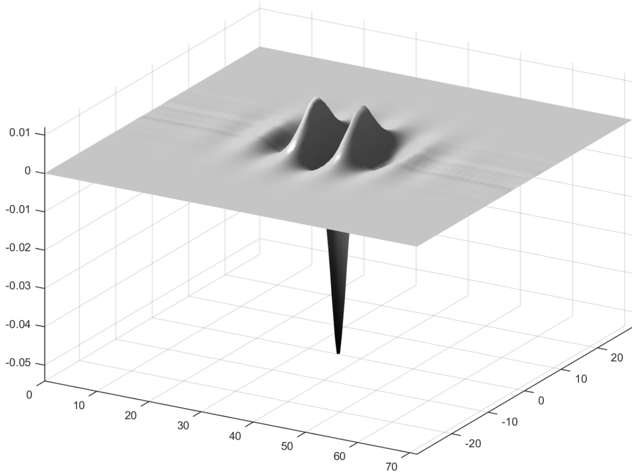

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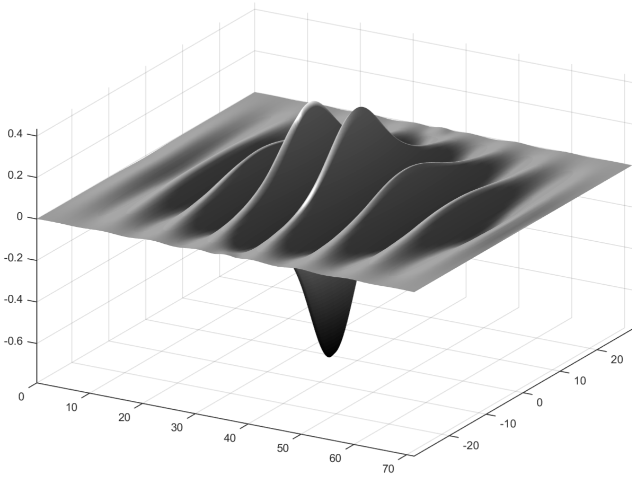
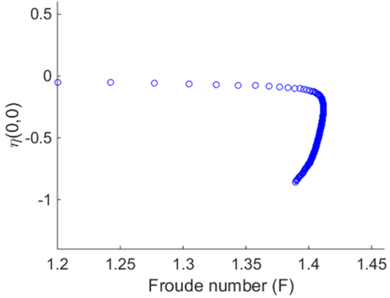


Summary of Bifurcation Branch

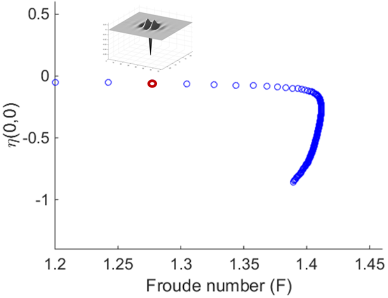
Forced depression waves using the nonlinear model for ice



Solutions in the red region are truncated, but after the turning point, obtain solitary lumps.

Summary of Bifurcation Branch

Forced depression waves using the nonlinear model for ice



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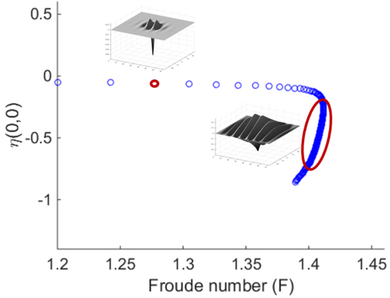

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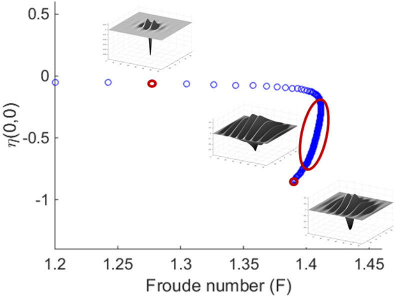

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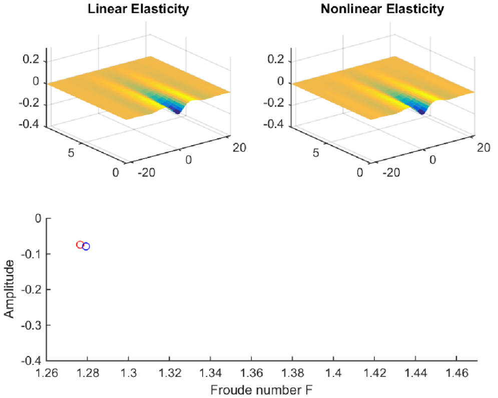
Bifurcation Branch

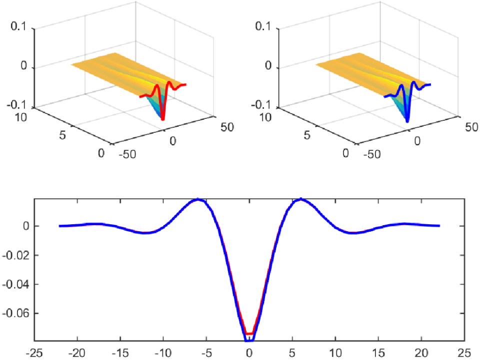
Comparison of the bifurcation branches for flexural-gravity waves with the linear and the nonlinear elasticity models



Note: both models give the same wave amplitude, but different Froude numbers

Flexural-Gravity Wave Profiles

Comparison of the solution profiles for linear elasticity model and the nonlinear elasticity model.



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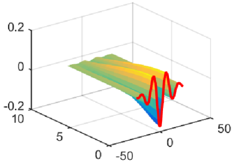

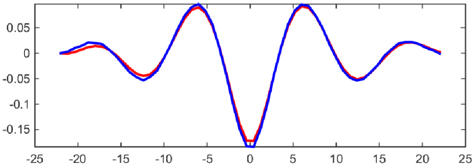

Flexural-Gravity Bifurcation Branch

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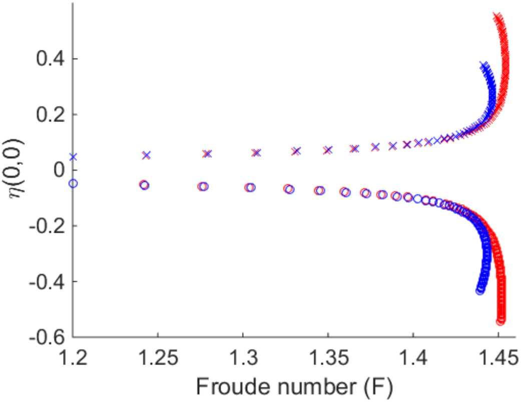

Elevation waves are represented as crosses and depression waves as circles.

Outline

Conclusions

- Can compute solutions to both models for flexural-gravity waves in 2D (periodic) and 3D (solitary)
- Both models produce similarly shaped profiles, but at different Froude numbers (or different wave speeds)
- The code is easy to use and easy to modify
- A variety of numerical methods have been tested

Future Work

- Examine the convergence to solitary lumps
- Compute accurate free surface waves without a forcing
- Do free surface depression or elevation waves bifurcate away from 0?

Thank you for your attention